Generation of exact solutions in cosmology on the basis of five-dimensional Projective Unified Field Theory

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Abstract. The homogeneous and isotropic model of the Universe is considered in the framework of the five-dimensional Projective Unified Field Theory in which the gravitation is described both by curvature of space-time and some hypothetical scalar field (σ-field). We propose a generation method of exact solutions. New exact Friedmann-like solutions for the dust model and inflationary solutions are found. It is shown that in the framework of the exponential type inflation we obtain a natural explanation why at present time we do not observe the σ-field effects or why these effects are so negligible.

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1 Introduction

In this work the homogeneous and isotropic model of the Universe is considered in the framework of the 5-dimensional Projective Unified Field Theory (PUFT) developed by E. Schmutzer [1, 2, 3]. In PUFT the gravitation is described both by curvature of space-time and by some hypothetical scalaric field (σ-field). To characterize the scalar field predicted in PUFT as a new fundamental phenomenon in Nature E. Schmutzer introduced the notion "scalarism" (adjective: scalaric) in analogy to electromagnetism. A source of this scalaric field can be both electromagnetic field and new attribute of matter named by E.Schmutzer scalaric mass.

The PUFT is based on the postulated 5-dimensional Einstein-like field equations. By projecting them into the 4-dimensional space-time one can obtain the following 4-dimensional field equations (the cosmological term is omitted here) [2]:

$$R_{mn} - \frac{1}{2}\, g_{mn}\, R = \kappa_0 \left(E_{mn} + \Sigma_{mn} + \Theta_{mn} \right) \tag{1}$$

is the generalized gravitational field equations (here $R_{mn} = R^{i}{}_{mni}$);

$$\text{a)} \quad H^{mn}{}_{;n} = \frac{4\pi}{c} j^m, \quad \text{b)} \quad B_{mn,k} + B_{km,n} + B_{nk,m} = 0, \quad \text{c)} \quad H_{mn} = e^{3\sigma} B_{mn} \tag{2}$$

is the generalized electromagnetic field equations;

$$\sigma^{,k}{}_{;k} = \kappa_0 \left(\frac{2}{3}\vartheta + \frac{1}{8\pi} B_{ik} H^{ik} \right) \tag{3}$$

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is the scalaric field equation. Here R_{mn} is the Ricci tensor,

$$E_{mn} = \frac{1}{4\pi}\left(B_{mk}H^k{}_n + \frac{1}{4}g_{mn}B_{ik}H^{ik}\right) \tag{4}$$

is the electromagnetic energy-momentum tensor,

$$\Sigma_{mn} = -\frac{3}{2\kappa_0}\left(\sigma_{,m}\sigma_{,n} - \frac{1}{2}g_{mn}\sigma_{,k}\sigma^{,k}\right) \tag{5}$$

is the scalaric energy-momentum tensor, Θ_{mn} is the energy-momentum tensor of the nongeometrized matter (substrate), H_{mn} and B_{mn} are respectively the electromagnetic induction and the field strength tensor, j^k is the electric current density, ϑ is the scalaric substrate density, $\kappa_0 = 8\pi G/c^4$ is the Einstein's gravitational constant (G is the Newton's gravitational constant). Latin indices run from 1 to 4; comma and semicolon denote partial and covariant derivatives respectively; signature of the metric is equal to $+2$.

These field equations lead to the following generalized energy conservation law and the continuity equation for electric current density respectively:

$$\text{a)}\quad \Theta^{mn}{}_{;n} = -\frac{1}{c}B^m{}_k j^k + \vartheta\sigma^{,m}, \qquad \text{b)}\quad j^m{}_{;m} = 0. \tag{6}$$

It should be noted that recently E.Schmutzer had offered a new variant of PUFT (see [3] and references quoted therein) with slightly different 4-dimensional field equations in comparison with the above-stated ones (one can find a detailed analysis of the geometric axiomatics of PUFT in [4]). The both variants are physically acceptable and deserve a comprehensive investigation. The analysis of equations (1) — (5) shows that all subsequent reasonings can be easily extended on the last version of PUFT.

2 Cosmological equations and generation of exact solutions

Let us consider a homogeneous and isotropic cosmological model with the Robertson-Walker metric in the well-known form:

$$\mathrm{d}s^2 = R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right] - c^2\mathrm{d}t^2, \tag{7}$$

where $R(t)$ is the scale factor and the constant of spatial curvature k takes values 0 or ±1.

For an electrically neutral continuum which is described by the energy-momentum tensor of a perfect fluid the field equations (1), (3) in the metric (7) lead us to the following system of differential equations (here dots denote time derivatives, ϱ is the mass density, p is the pressure):

$$\frac{\ddot{R}}{R} = -\frac{\kappa_0 c^2}{6}\left(\varrho c^2 + 3p\right) - \frac{1}{2}\dot{\sigma}^2, \tag{8}$$

$$\frac{\ddot{R}}{R} + \frac{2\left(\dot{R}^2 + kc^2\right)}{R^2} = \frac{\kappa_0 c^2}{2}\left(\varrho c^2 - p\right), \tag{9}$$

$$\ddot{\sigma} + 3\frac{\dot{R}}{R}\dot{\sigma} = -\frac{2}{3}\kappa_0 c^2\vartheta, \tag{10}$$

while the generalized energy conservation law (6) gives

$$\dot{\varrho} + 3\frac{\dot{R}}{R}\left(\varrho + \frac{p}{c^2}\right) = \frac{\vartheta}{c^2}\dot{\sigma}. \tag{11}$$

The equations (8) to (11) determine the dynamics of the cosmological model if the equations of state, i.e. $p = p(\varrho)$ and $\vartheta = \vartheta(\varrho)$, are given. The Friedmann model corresponds to the special case $\vartheta = 0$ and $\dot{\sigma} = 0$ of our model. Unfortunately, the above system of differential equations leads to an Abel differential equation [5] and till now was solved exactly only for some special cases [5, 6, 7, 8].

In order to obtain new exact solutions let us consider a problem under a slightly different aspect. Firstly, we shall use an arbitrariness in a choice of an equation of state $\vartheta = \vartheta(\varrho)$, because the functional form $\vartheta(\varrho)$ is not determined within the theory. Secondly, notice that above equations are not independent. For instance, it is possible to show that the last equation (11) is the differential consequence of (8) to (10). Therefore at the further treatment we shall use only the equations (8) to (10). If the equations of state for ordinary matter is given: $p = \nu\varrho c^2$ $(-1 \le \nu < 1)$, then the equations (8) to (10) can be reduced to the form where the functions $\sigma(t)$, $\varrho(t)$ and $\vartheta(t) \equiv \vartheta(\varrho(t))$ are expressed through the function $R(t)$ and their derivatives:

$$\sigma(t) = \pm\sqrt{\frac{8}{3(1-\nu)}}\int\sqrt{-\frac{\ddot{R}}{R} - \frac{1+3\nu}{2}\,\frac{\dot{R}^2 + kc^2}{R^2}}\,\mathrm{d}t + \sigma_0\,, \tag{12}$$

$$\varrho(t) = \frac{2}{(1-\nu)\kappa_0 c^4}\left[\frac{\ddot{R}}{R} + \frac{2\left(\dot{R}^2 + kc^2\right)}{R^2}\right], \tag{13}$$

$$\vartheta(t) = -\frac{3}{2\kappa_0 c^2 R^3}\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\sigma}R^3\right). \tag{14}$$

Here σ_0 is a constant of integration and $\nu \neq 1$ (it is necessary to consider the case $\nu = 1$ separately). If we define the functional dependence of the scale factor on time $R(t)$, we can find the corresponding functions $\sigma(t)$, $\varrho(t)$ and $\vartheta(t)$. This will ensure the necessity of the chosen scenario of the Universe's evolution. However, the selection of the dependence $R(t)$ is not free, because the natural requirement $\varrho \ge 0$ with taking account of (12), (13) gives the following limitation on the choice of scale factor:

$$-\frac{\ddot{R}}{R} - \frac{1+3\nu}{2}\,\frac{\dot{R}^2 + kc^2}{R^2} \ge 0, \tag{15}$$

$$\frac{\ddot{R}}{R} + \frac{2\left(\dot{R}^2 + kc^2\right)}{R^2} \ge 0. \tag{16}$$

It is obvious that the pair of functions (13) and (14) define the parametric dependence $\vartheta = \vartheta(\varrho)$, which in some cases can be reduced to an explicit form by elimination t.

It should be noted that a similar method was proposed in [9, 10], where the idea that the shape of the potential of self-interacting scalar field in standard inflationary models is not fixed, allows to obtain new exact solutions for inflation (in this context see also [11, 12]). In [9] this approach was called the method of "fine turning of the potential".

3 Examples

3.1 Exact solutions for the dust model

To illustrate the above-stated method, we shall give a series of simple examples. We start from the case of dust model ($p = 0$ or constant $\nu = 0$). Let us consider a power-law behavior of the scale factor:

$$R(t) = At^n \quad (1/3 \le n \le 2/3), \tag{17}$$

where A, n are positive constants and the limiting condition for n was obtained by means of (15) and (16). It is easy to show that (17) satisfies (15) and (16) for $k = 0$ always and for $k = \pm 1$ only for a restricted period of time. Further, for simplicity we restrict our consideration to the spatially-flat model ($k = 0$). Taking account of (17) the equations (12), (13) and (14) allow us to find the exact solutions for $\sigma(t)$, $\dot\sigma(t)$, $\varrho(t)$ and $\vartheta(t)$:

$$\sigma(t) = \pm\, 2\,\sqrt{(2n - 3n^2)/3}\ \ln t + \sigma_0\ , \tag{18}$$

$$\dot\sigma(t) = \pm\, 2\,\sqrt{(2n - 3n^2)/3}\ \frac{1}{t}\ , \tag{19}$$

$$\varrho(t) = \frac{2(3n^2 - n)}{\kappa_0 c^4}\ \frac{1}{t^2}\ , \tag{20}$$

$$\vartheta(t) = \mp\ \frac{(3n-1)\sqrt{3(2n - 3n^2)}}{\kappa_0 c^2}\ \frac{1}{t^2}\ . \tag{21}$$

It is interesting to note that the case $n = 1/3$ corresponds to a model of the Universe only with scalaric σ-field (for this case the exact solutions were found earlier in [8]). And the case $n = 2/3$ corresponds to the standard Einstein-de Sitter model of Friedmann cosmology. The equations (20) and (21) allow to obtain the equation of state $\vartheta = \vartheta(\varrho)$ in explicit form:

$$\vartheta = \mp\ \frac{\sqrt{3(2n - 3n^2)}}{2n}\ \varrho c^2 \quad (n \ne 1/3)\ , \tag{22}$$

and also to find the following range for ϑ: $0 \le |\vartheta| < 3\varrho c^2$.

Let us note here the simplest consequences of the considered model for observational cosmology. The Hubble parameter and the age of the Universe are given by

$$H(t) \equiv \frac{\dot R}{R} = \frac{n}{t}\ ; \qquad t_0 = \frac{n}{H_0}\ , \quad \frac{1}{3H_0} \le t_0 \le \frac{2}{3H_0}\ , \tag{23}$$

where subscript zero denotes present values. For the deceleration parameter q_0 we have

$$q_0 = q \equiv -\frac{\ddot R R}{\dot R^2} = \frac{1-n}{n}\ , \quad 1/2 \le q_0 \le 2\ . \tag{24}$$

The parameter λ_0 (already introduced in [13]), characterizing the scalaric field, is given by

$$\lambda_0 \equiv \frac{1}{H_0}\frac{\mathrm{d}\sigma(t_0)}{\mathrm{dt}} = \pm\, 2\,\sqrt{\frac{1}{n}\left(\frac{2}{3} - n\right)}\ , \quad 0 \le |\lambda_0| \le 2\ . \tag{25}$$

It should be noted that the parameter λ_0 is, in principle, a measurable quantity [13]. Also, for the flat model considered here, λ_0 and q_0 are not independent parameters. They are related to each other by means of the identity $2q_0 - 3{\lambda_0}^2/4 = 1$, while the density parameter is given by $\Omega_0 = 1 - {\lambda_0}^2/4$ and $\Omega_0 < 1$ (in more detail see [13, 14]).

3.2 Exponential type inflation

Let us consider a cosmological model in which the contribution of vacuum energy prevails in the total energy density so that the equation of state as $p = -\varrho c^2$ is realized (see e.g. [15]). In this case we suppose that the scale factor $R(t)$ increases very fast according to the exponential law like in the classical inflation:

$$R(t) = Ae^{Ht}, \tag{26}$$

where A and H are positive constants. Moreover, here the H plays the role of the Hubble constant, because $\dot{R}/R = H$ for any time. The expression (26) satisfies the equations (15) and (16) for $k = 0$ and $k = 1$, but one cannot be satisfied if $k = -1$.

If $k = 0$, the substitution (26) in (12) to (14) gives the simple solution:

$$\sigma = \text{const } (\dot{\sigma} = 0) \ , \quad \vartheta = 0 \ , \quad \varrho = \frac{3H^2}{\kappa_0 c^4} = \text{const} \ , \tag{27}$$

which coincides with the classical de-Sitter solution of the General Relativity Theory.

If $k = 1$, then from (12) to (14) we find:

$$\sigma(t) = \pm \frac{2c}{\sqrt{3}AH} \left(1 - e^{-Ht}\right) + \sigma_0 \ , \tag{28}$$

$$\dot{\sigma}(t) = \pm \frac{2c}{\sqrt{3}AH} \ e^{-Ht} \ , \tag{29}$$

$$\varrho(t) = \frac{3H^2}{\kappa_0 c^4} \left(1 + \frac{2c^2}{3A^2H^2} \ e^{-Ht}\right) \ , \tag{30}$$

$$\vartheta(t) = \mp \frac{2\sqrt{3} \ H}{\kappa_0 c A} \ e^{-Ht} \ . \tag{31}$$

Notice that the obtained solutions asymptotically tend to (27), if t is large. Consequently, together with the exponential growth of scale factor the scalaric field effects become more and more negligible, so that at large t the considered model passes over to the standard inflationary de-Sitter model. It is interesting to note that within this simplest inflationary model we get a natural explanation why at present time we do not observe the scalaric field effects or why these effects are so small.

Let us indicate here another exponential type of solutions possessing such properties. Such a solution corresponding to expansion without a singularity is given by

$$R(t) = A\cosh \omega t, \tag{32}$$

$$\sigma(t) = \pm \frac{2a}{\sqrt{3}\omega} \arctan\left(\sinh \omega t\right) + \sigma_0 \ , \tag{33}$$

$$\dot{\sigma}(t) = \pm \frac{2a}{\sqrt{3}} \ \frac{1}{\cosh \omega t} \ , \tag{34}$$

$$\varrho(t) = \frac{3\omega^2}{\kappa_0 c^4} \left(1 + \frac{2a^2}{3\omega^2} \ \frac{1}{\cosh^2 \omega t}\right) \ , \tag{35}$$

$$\vartheta(t) = \mp \frac{2\sqrt{3}a\omega}{\kappa_0 c^2} \frac{\sinh \omega t}{\cosh^2 \omega t} , \tag{36}$$

where $a \equiv \sqrt{kc^2 A^{-2} - \omega^2}$, and A, ω are positive constants. In this case the conditions (15) and (16) can only be satisfied if $k = 1$.

Next, one can find the solution corresponding to expansion from a singularity:

$$R(t) = A \sinh \omega t, \tag{37}$$

$$\sigma(t) = \pm \frac{a}{\sqrt{3}\omega} \ln \frac{\cosh \omega t - 1}{\cosh \omega t + 1} + \sigma_0 , \tag{38}$$

$$\dot{\sigma}(t) = \pm \frac{2a}{\sqrt{3}} \frac{1}{\sinh \omega t} , \tag{39}$$

$$\varrho(t) = \frac{3\omega^2}{\kappa_0 c^4} \left(1 + \frac{2a^2}{3\omega^2} \frac{1}{\sinh^2 \omega t} \right) , \tag{40}$$

$$\vartheta(t) = \mp \frac{2\sqrt{3}a\omega}{\kappa_0 c^2} \frac{\cosh \omega t}{\sinh^2 \omega t} , \tag{41}$$

where now $a \equiv \sqrt{kc^2 A^{-2} + \omega^2}$. In this case (15) and (16) are automatically true for $k = 1$ and $k = 0$, and can always be satisfied if $k = -1$. It should be noted that there is one more solution, corresponding to harmonic behavior of the scale factor:

$$R(t) = A \sin \omega t, \tag{42}$$

$$\sigma(t) = \pm \frac{a}{\sqrt{3}\omega} \ln \frac{\cos \omega t - 1}{\cos \omega t + 1} + \sigma_0 , \tag{43}$$

$$\dot{\sigma}(t) = \pm \frac{2a}{\sqrt{3}} \frac{1}{\sin \omega t} , \tag{44}$$

$$\varrho(t) = \frac{3\omega^2}{\kappa_0 c^4} \left(-1 + \frac{2a^2}{3\omega^2} \frac{1}{\sin^2 \omega t} \right) , \tag{45}$$

$$\vartheta(t) = \mp \frac{2\sqrt{3}a\omega}{\kappa_0 c^2} \frac{\cos \omega t}{\sin^2 \omega t} . \tag{46}$$

This is so-called [12] trigonometric counterpart to the solution (37) to (41).

3.3 Power law inflation

As a further example, consider a power law behavior of the scale factor:

$$R(t) = At^m \quad (m \geq 1), \tag{47}$$

where A and m are positive constants. The relation (47) always satisfies the equations (15) and (16) for $k = 0$ and for $k = 1$, and can be satisfied for a restricted period of time if $k = -1$.

If $k = 0$, the equations (12) to (14) give the simple solution:

$$\sigma(t) = \pm\, 2\, \sqrt{\frac{m}{3}}\; \ln t + \sigma_0\,, \tag{48}$$

$$\dot{\sigma}(t) = \pm\, 2\, \sqrt{\frac{m}{3}}\; \frac{1}{t}\,, \tag{49}$$

$$\varrho(t) = \frac{3m^2 - m}{\kappa_0 c^4}\; \frac{1}{t^2}\,, \tag{50}$$

$$\vartheta(t) = \mp\, \frac{\sqrt{3m}(3m-1)}{\kappa_0 c^2}\; \frac{1}{t^2}\,. \tag{51}$$

In this case the equation of state $\vartheta = \vartheta(\varrho)$ in explicit form is given by $\vartheta = \mp\, \sqrt{3/m}\; \varrho c^2$.

If $k = \pm 1$, then we have $(m \neq 1)$

$$\sigma(t) = \pm\, \frac{\sqrt{m}}{3(1-m)} \left[2\sqrt{1 + at^{-2m+2}} + \ln \frac{\sqrt{1 + at^{-2m+2}} - 1}{\sqrt{1 + at^{-2m+2}} + 1} \right] + \sigma_0\,, \tag{52}$$

$$\dot{\sigma}(t) = \pm\, 2\, \sqrt{\frac{m}{3}}\; \frac{\sqrt{1 + at^{-2m+2}}}{t}\,, \tag{53}$$

$$\varrho(t) = \frac{1}{\kappa_0 c^4}\; \frac{3m^2 - m + 2mat^{-2m+2}}{t^2}\,, \tag{54}$$

$$\vartheta(t) = \mp\, \frac{\sqrt{3m}}{\kappa_0 c^2}\; \frac{3m - 1 + 2mat^{-2m+2}}{t^2\, \sqrt{1 + at^{-2m+2}}}\,, \tag{55}$$

where now $a \equiv kc^2 m^{-1} A^{-2}$. It is easy to see that the obtained solution tend asymptotically to (48) to (51), if t is large. Let us also note that this family of solutions is like a standard non-inflationary big-bang if $1/3 \leq m < 1$. In the case of $m = 1$ we have linear inflation. The corresponding solution is given by

$$R(t) = At, \tag{56}$$

$$\sigma(t) = \pm\, 2\, \sqrt{(1+a)/3}\; \ln t + \sigma_0\,, \tag{57}$$

$$\dot{\sigma}(t) = \pm\, 2\, \sqrt{(1+a)/3}\; \frac{1}{t}\,, \tag{58}$$

$$\varrho(t) = \frac{2(1+a)}{\kappa_0 c^4}\; \frac{1}{t^2}\,, \tag{59}$$

$$\vartheta(t) = \mp\, \frac{2\, \sqrt{3(1+a)}}{\kappa_0 c^2}\; \frac{1}{t^2}\,, \tag{60}$$

where now $a \equiv kc^2 A^{-2}$.

4 Conclusions

In the present paper, using the arbitrariness in the choice of an equation of state for the scalaric substrate energy density $\vartheta = \vartheta(\varrho)$ which is not determined within the PUFT, we have proposed a generation method of exact solutions for the homogeneous and isotropic models of the Universe. This method has allowed us to find new Friedmann-like solutions for the dust model as well as the solutions for the simplest inflationary models. It is interesting to note that within the framework of the exponential type of inflation we have received a natural explanation why at present time we do not observe the scalaric field effects or why these effects are so negligible. It should be noted that a cosmological model, in which the σ-field plays an essential role both at early stages of the Universe's evolution and at present, was considered in detail in recent papers of E.Schmutzer [16, 17].

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